ASYMPTOTIC THEORY OF LINEAR DIFFERENTIAL EQUATIONS SINGULAR IN SEVERAL PARAMETERS

BY

CHARLES WILLIAM MORAN

B.S., Lewis Institute, 1933

A.M., Loyola University, 1936

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1940

URBANA, ILLINOIS

1940

ASYMPTOTIC THEORY OF LINEAR DIFFERENTIAL EQUATIONS SINGULAR IN SEVERAL PARAMETERS

BY

CHARLES WILLIAM MORAN

B.S., Lewis Institute, 1933

A.M., Loyola University, 1936

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1940

URBANA, ILLINOIS

1940

ASYMPTOTIC THEORY OF LINEAR DIFFERENTIAL EQUATIONS SINGULAR IN SEVERAL PARAMETERS

I. Introduction

The purpose of this paper is the consideration of the character of solutions of equations of the type

$$(1)\quad L_n(x,\lambda;y) = \sum_{k=0}^{n} b_{n-k}(x,\lambda_1\ldots\lambda_m)y^{[k]}(x,\lambda_1\ldots\lambda_m) = 0^{\P}$$

The $b_i(x,\lambda_1\ldots\lambda_m)$ are assumed to be indefinitely differentiable in x for $(a\leq x\leq b)$ and analytic in the λ_i's for $|\lambda_i|\geq\rho_i > 0, (i=1,\ldots m)$, the λ_i's being parameters. The coefficients of (1) are representable by convergent series of the form

$$(2)\quad \alpha_i(x,\lambda) = \lambda_1^{{}_iB_1}\ldots\lambda_m^{{}_iB_m}\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_m=0}^{\infty}\alpha_{i;\mu_1\ldots\mu_m}(x)\lambda_1^{-\mu_1}\ldots\lambda_m^{-\mu_m}.$$

The ${}_iB_j$ $(i=1,\ldots n; j=1,\ldots m)$ are positive integers or zero and the $\alpha_{i;\mu_1\ldots\mu_m}(x)$ are indefinitely differentiable on (a,b), $|\lambda_i|\geq\rho_i$. More generally, the $b_i(x,\lambda_1\ldots\lambda_m)$ may be merely asymptotic (§2) to a finite or infinite number of terms to such series (possibly divergent), when $a\leq x\leq b$ and the λ_i are in a suitable region R_i extending to infinity.

In general, the coefficients $b_i(x,\lambda)$ have poles at $\lambda_i=\infty$ $(i=1,\ldots m)$ and the problem may be described as an m-fold singular point problem.

Linear equations of the n-th order have received a great deal of attention; equations of the type (1) have been studied by W.J.Trjitzinsky and the method of Iterations, developed by him (3) is used here. For additional bibliography and history of the problem the paper by W.J.Trjitzinsky (5) may be consulted. Other work in this direc-

¶ $y^{[k]} = \frac{\partial^k y}{\partial x^k}$

tion has been done by C.C.Hurd for the case of one parameter, singular in the parameter and in the variable of integration(4), and in the case of two parameters(2).

Notations.

$$\phi(x,\lambda_1\cdots\lambda_m)=\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_m=0}^{\infty}a_{\mu_1\cdots\mu_m}(x)\lambda_1^{\mu_1}\cdots\lambda_m^{\mu_m}=\sum_{\mu=0}^{\infty}a_\mu(x)\lambda^\mu=\phi(x,\lambda).$$

In this notation the subscript (or exponent) may be considered as an m-tuple number $\mu=(\mu_1,\ldots\mu_m)$ where the μ_i's range over the set F of positive integers and zero. The ordinary laws of addition and multiplication by a number in F hold, and in addition the following convention; $\mu>\nu$ implies $\mu_i>\nu_i$ $(i=1,\ldots m)$. The unit will be designated in the usual way, e_i representing a number with unity in the i-th place and zero in the others. When a is an integer $(a)=(a,\ldots a)$, when $a=0$, and in cases where no ambiguity will result, the parentheses will be omitted. Also $\lambda^\mu=\lambda_1^{\mu_1}\lambda_2^{\mu_2}\cdots\lambda_m^{\mu_m}$.

Definitions.

1. Asymptotic relations. Let R_i be a region extending to infinity in the λ_i plane, and (a,b) a real interval. Let $\phi(x,\lambda)$ be a function defined for $(a\leq x\leq b)$ and λ_i in R_i $(i=1,\ldots m)$. Having given a series, divergent or convergent

$$(3)\quad \sigma(x,\lambda)=\lambda^\theta\sum_{\mu=0}^{\infty}\sigma(x)\lambda^{-\mu}\qquad(\beta_i>0,(i=1,\ldots m)$$

we shall say that the $\phi(x,\lambda)$ is asymptotic to $\sigma(x,\lambda)$ to (w) terms provided that

$$(4)\quad \phi(x,\lambda)=\lambda^\theta\sum_{\mu=0}^{w-1}\sigma(x)\lambda^{-\mu}+\sum_{i=1}^{m}\lambda^{w_i}c_{w_i}(x,\lambda).$$

Here the $c_{w_i}(x,\lambda)$ are defined for λ_i in R $(i=1,\ldots m)$ and $(a\leq x\leq b)$ such that

$$(4a)\quad |c_{w_i}(x,\lambda)|\leq c_{w_i}$$

Relations (4), (4a) will be written

(4b) $\phi(x,\lambda) \sim \sigma(x,\lambda)$

2. A series of the form (3) in which the $\sigma_\mu(x)$ are indefinitely differentiable will be referred to in the sequel as a σ-series. The equation (1) can now be written (using the above notation) in the form

$$(A)\quad L_n(x,\lambda;y) = \sum_{k=0}^{n} \lambda^{(n-k)H} a_{n-k}(x,\lambda) y^{[k]}(x,\lambda) = 0$$

$$a_0(x,\lambda) = 1; \quad a_n(x,\lambda) \neq 0$$

where the $a_{n-k}(x,\lambda)$ are convergent series of the form

$$(A^*)\quad \alpha_{n-k}(x,\lambda) = \sum_{\mu=0}^{\infty} a_{n-k,\mu}(x)\, \lambda^{-\mu}$$

or are asymptotic to such possibly divergent series. The H_i's are integers taken as small as possible, $H_i \geq 1$ $(i=1,\ldots m)$.

Associated with equation (A) is the system

$$(B)\quad Y^{[1]}(x,\lambda) = Y(x,\lambda)A(x,\lambda) \text{ ¶}$$

where $Y(x,\lambda) = (y_{ij}(x,\lambda)),(i=1,\ldots n)$ denotes a matrix. It is obvious that if $(y_{ij}(x,\lambda))$ is a matrix solution of (B), then

$$(5)\quad (y_{ij}(x,\lambda)) = (y_{i1}^{[j-1]}(x,\lambda))$$

and the $y_{i1}(x,\lambda)$ (first column) will form a fundamental set of solutions of (A). Conversely, if a set of functions $y_i(x,\lambda)$ forms a full set of solutions of (A) then the matrix $Y(x,\lambda) = (y_i^{[j-1]}(x,\lambda))$ will satisfy (B).

II. Formal Series.

We now show that there exists a fundamental set of solutions which formally

¶ $Y^{[1]}(x,\lambda) = \frac{\partial}{\partial x} Y(x,\lambda) = (\frac{\partial}{\partial x} y_{ij}(x,\lambda))$.

satisfy equation (A). The characteristic equation associated with (A) is

$$(C) \quad E(x,\rho) = \sum_{k=0}^{n} a_{n-k,0}(x)\rho^k = 0.$$

We consider only the case in which the n roots of (C) are distinct. (C) cannot have a zero root since in (A), $a_n(x,\lambda) \neq 0$.

We say that (formally)

$$(1) \quad s_i(x,\lambda) = e^{\lambda^H Q_i(x,\lambda)}\, {}_i\sigma(x,\lambda)$$

where

$$(1a) \quad Q_i(x,\lambda) = \sum_{\mu=0}^{H-1} {}_iq_\mu(x)\lambda^{-\mu}$$

$$(1b) \quad {}_i\sigma(x,\lambda) = \sum_{\mu=0}^{\infty} {}_i\sigma_\mu(x)\lambda^{-\mu}$$

is a solution of (A) if upon substituting (1) in (A) we obtain relationships involving the $a_i(x,\lambda)$ from which the ${}_iq_\mu(x,\lambda)$ and the ${}_i\sigma_\mu(x,\lambda)$ may be determined. We shall also determine conditions under which the $s_i(x,\lambda)$ (i=1,...n) form a fundamental set of solutions.

Substituting (1) in (A) we have formally

$$(2) \quad L_n(x,\lambda;s(x,\lambda)) = \sum_{k=0}^{n} \lambda^{(n-k)H} a_{n-k}(x,\lambda)s^{[k]}(x,\lambda) = 0.$$

Using the formula[1] for the derivative of a function of a function gives

$$(3) \quad \sum_{k=0}^{n} a_{n-k}(x,\lambda) \sum_{\ell=0}^{k} \binom{k}{\ell} \sigma^{[\ell]}(x,\lambda)P_{k-\ell}(x,\lambda) = 0.$$

where

$$(3a) \quad P_{k-\ell}(x,\lambda) = \left\{ (Q^{[1]})^{k-\ell}\lambda^{-H\ell} + \frac{(k-\ell)!}{(k-\ell-2)!2!}(Q^{[1]})^{k-\ell-2}Q^{[2]}\lambda^{-H(\ell+1)} + \cdots + Q^{[k-\ell]}\lambda^{-H(k-1)} \right\}.$$

---

[1] F. Faa' di Bruno, Sullo Sviluppo Delle Funzioni Annali di Scienze Matematiche e Fisiche, vol.6, (1855) p.479.

We now equate coefficients of $\lambda^{-\gamma}$ in (3) for ($\gamma=1,2,\ldots$), which gives for $\gamma < H$

$$(4)\quad \sum_{k=0}^{n} a_{n-k,0}(x)\left[\sum_{\gamma} q_{\gamma}^{[i]}(x)\right]^{k} = -\sum_{k=0}^{n} a_{n-k,\gamma}(x)\left[q_{0}^{[i]}(x)\right]^{k}$$

where γ is the exponent of λ^{-1} in

$$\left[\sum_{\mu=0}^{H-1} q_{\mu}^{[i]}(x)\,\lambda^{-\mu}\right]^{k}$$

from which the $q_{\gamma}(x)$ can be determined in order. For the case of $\nu=\omega=\mu=0$, we have

$$(5)\quad \sum_{k=0}^{n} a_{n-k,0}(x)\left[q_{0}^{[i]}(x)\right]^{k} = 0.$$

This shows that the $q_{0}^{[i]}(x)$ are roots of equation (C), and that there are n distinct non-constant values for $q_{0}(x)$. Values of $q_{\mu}(x)$ between $q_{(\alpha)}(x)$ and $q_{(\alpha+1)}(x)$ can be determined by considering in (4) values of $\gamma = \alpha + e_{i}$ ($i=1,\ldots m$). The above equations determine completely the n distinct values of $Q_{i}(x,\lambda)$, ($i=1,\ldots n$).

Equating coefficients of $\lambda^{-\gamma}$ for $\gamma \leq H$ in (3), noting that the $P_{k-l}(x,\lambda)$ are completely determined, we obtain, in order, the values of $\sigma_{i}(x)$, ($i=1,\ldots$).

LEMMA I. Given the equation (A) with coefficients of the form (A*). Assuming that there are no multiple roots of the characteristic equation (C), there exists a full set of formal solutions of this equation, of the form

$$s_{i}(x,\lambda) = e^{\lambda^{H} Q_{i}(x,\lambda)}\,{}_{i}\sigma(x,\lambda)$$

where the $Q_{i}(x,\lambda)$ and the ${}_{i}\sigma(x,\lambda)$ are given by (1a) and (1b). The ${}_{i}q_{\mu}(x)$ are indefinitely differentiable for ($a \leq x \leq b$) and are determined from equation (4), moreover, the ${}_{i}q_{0}^{[i]}(x)$ are roots of equation (C). The ${}_{i}\sigma_{\mu}(x)$ are indefinitely differentiable on ($a \leq x \leq b$) and are determined from equation (3).

III. Integrations.

Let R_i be a region in the λ_i plane ($i=1,\dots m$) such that for x in (c,d), a subinterval of (a,b) in which $q_o^{[i]}(x)\neq 0$, and λ_i in R_i, either

(D) $R\tilde{Q}^{[i]} \geqq 0$[¶] or (E) $R\tilde{Q}^{[i]} \leqq 0$, $\tilde{Q} = \lambda^H Q(x,\lambda)$

holds. We consider the equation

(1) $y^{[i]}(x,\lambda) = e^{\lambda^H Q(x,\lambda)} a(x,\lambda)$

where $Q(x,\lambda)$ is as given in (1a)§2. The function $a(x,\lambda)$ is analytic in each λ_i and satisfies

(1a) $a(x,\lambda) \sim \sigma(x,\lambda)$

where $\sigma(x,\lambda)$ is as given in (1b)§2. The integral coordinates of w are assumed to be suitably great ($w \geqq H$).

The formal equation associated with (1) is

(2) $y^{[i]}(x,\lambda) = e^{\lambda^H Q(x,\lambda)} \sigma(x,\lambda)$

where $Q(x,\lambda)$ and $\sigma(x,\lambda)$ are as given above. Substituting

(3) $y(x,\lambda) = e^{\lambda^H Q(x,\lambda)} \lambda^{-H} \eta(x,\lambda)$

in (2) shows that (2) is formally satisfied by (3) where $\eta(x,\lambda)$ is given by

(4) $\eta(x,\lambda) = \sum_{\mu=0}^{\infty} \eta_\mu(x)\, \lambda^{-\mu}$

(4a) $\eta_r(x) = \dfrac{\sigma_r(x) - \sum_{\nu}^{r} \eta_\nu(x)\, q_{r-\nu}^{[i]}(x)}{q_0^{[i]}(x)}$ $\qquad r \leqq H$

(4b) $\eta_r(x) = \dfrac{\sigma_r(x) - \eta_{r+H}(x) - \sum_{\nu}^{r} \eta_\nu(x)\, q_{r-\nu}^{[i]}(x)}{q_0^{[i]}(x)}$ $\qquad r > H$

[¶] $RQ(x,\lambda)$ = Real part of $Q(x,\lambda)$.

In both (4a) and (4b), the $\sum_r$ in the numerator indicates summation over all of the solutions of $\nu+\mu=r$ except $\nu=r$, $\mu=0$.

Form the function

$$(5) \qquad t(x,\lambda) = e^{\lambda^H Q(x,\lambda)} \lambda^{-H} \sum_{\mu=0}^{w-1} \eta_\mu(x)$$

and apply the transformation

$$(6) \qquad y(x,\lambda) = t(x,\lambda) + z(x,\lambda)$$

to equation (1). We find that

$$(7) \qquad z^{[1]}(x,\lambda) = e^{\lambda^H Q(x,\lambda)} \lambda^{-(w-1)} g_w(x,\lambda) \qquad |g_w(x,\lambda)| \leq g_w.$$

If we assume condition (D) to hold, we get

$$(8) \qquad z(x,\lambda) = e^{\widetilde{Q}(x,\lambda)} \lambda^{-(w-1)} z_w(x,\lambda) \qquad |z_w(x,\lambda)| \leq z_w$$

Rewriting (6) using (5) and (8) gives

$$(9) \qquad y(x,\lambda) = e^{\widetilde{Q}(x,\lambda)} \lambda^{-H} \left[\sum_{\mu=0}^{w-1} \eta_\mu(x) \lambda^{-\mu} + \lambda^{-w} b_w(x,\lambda) \right] \qquad w \geq 1$$

$$|b_w(x,\lambda)| \leq b_w \qquad b_w \text{ independent of } x \text{ and } \lambda$$

If we designate the formal solution $y(x,\lambda)$ as given in (3) by $s(x,\lambda)$, then (9) is equivalent to saying that $y(x,\lambda) \sim s(x,\lambda)$. Similiarly if condition (E) holds.

LEMMA II. The equation

$$y^{[1]}(x,\lambda) = e^{\lambda^H Q(x,\lambda)} a(x,\lambda)$$

is satisfied by a function $y(x,\lambda)$ in the regions defined by either of the conditions (D) or (E). This function $y(x,\lambda)$ is asymptotic to any suitable number of terms to the corresponding formal solution $s(x,\lambda)$

IV. Iterations.

We assume that there exist regions R_i extending to infinity in the λ_i plane ($i=1,\ldots m$) in which, when x is on (c,d) the following ordering is maintained

(1) $R\tilde{Q}_{\tau_1}^{[i]}(x,\lambda) = R\tilde{Q}_{\tau_2}^{[i]}(x,\lambda) = \ldots = R\tilde{Q}_{\tau_\nu}^{[i]}(x,\lambda) >$

$R\tilde{Q}_{\tau_{i+1}}^{[i]}(x,\lambda) = \ldots = R\tilde{Q}_{\tau_2}^{[i]}(x,\lambda) > R\tilde{Q}_{\tau_{i+1}}^{[i]}(x,\lambda) = .$

$\ldots = R\tilde{Q}_{\tau_3}^{[i]}(x,\lambda) > \ldots > R\tilde{Q}_{\tau_{\nu-1}+1}^{[i]}(x,\lambda) = \ldots\ldots$

$= R\tilde{Q}_{\tau_\nu}^{[i]}(x,\lambda) \qquad (1 \leq \tau_1 < \tau_2 < \ldots < \tau_{\nu-1} < \tau_\nu = n).$

With the λ_i's restricted to such regions we consider the matrix system

(2) $Y^{[i]}(x,\lambda) = Y(x,\lambda)\, B(x,\lambda).$

The elements of $B(x,\lambda)$ are analytic in the λ_i's and indefinitely differentiable in x and satisfy

(3) $b_{ij}(x,\lambda) \sim \beta_{ij}(x,\lambda) \qquad \beta_{ij}(x,\lambda) = \sum_{\mu=0}^{\infty} \beta_{ij\mu}(x)\lambda^{\beta-\mu}$

where the β_i's are positive integers and $|\beta_{ij}| \neq 0$.

Suppose that the formal system

(4) $S^{[i]}(x,\lambda) = S(x,\lambda)(\beta_{ij}(x,\lambda))$

has a matrix solution

(5) $S(x,\lambda) = (s_{ij}(x,\lambda)) = (e^{\lambda^H Q_i(x,\lambda)} \sigma_{ij}(x,\lambda))$

where the $\sigma_{ij}(x,\lambda)$ are σ series. Define

(6) $T(x,\lambda) \quad (e^{\tilde{Q}_i(x,\lambda)} \tau_{ij}(x,\lambda)) \qquad \tau_{ij}(x,\lambda) = \sum_{\mu=0}^{t-1} \sigma_{ij\mu}(x)\lambda^{-\mu}$

where the $t_1 \ldots t_m$ are suitably great integers. If the transformation

(7) $Y(x,\lambda) = Z(x,\lambda)\, T(x,\lambda)$

be applied to (2) there results

(8) $Z^{[i]}(x,\lambda) = Z(x,\lambda) C(x,\lambda).$

For x on (a,b), let $a=x_0\leq x_1\leq\ldots\leq x_\ell=x$, be a mode of subidvision of $(a.b)$ and write $v_k=x_k-x_{k-1}$ $(k=1,\ldots\ell)$ and choose the x_i so that the $\lim_{\ell\to\infty}$ max. $v_k=0$. Let $y_1,\ldots y_\ell$ be numbers such that $x_{k-1}\leq y_k\leq x_k$ $(k=1,\ldots\ell)$. Form the matrix

$$(9)\quad Z_\ell(x,\lambda)=(I+v_1C(y_1,\lambda))\ldots\ldots(I+v_kC(y_k,\lambda).$$

By the theory of product integrals*,

$$(10)\quad Z(x,\lambda)=\lim_{\ell\to\infty}Z_\ell(x,\lambda)=\int_a^{\widehat{x}}(C(x,\lambda)+I)dx$$

is a solution of equation (8). Using inequalities (1) on the elements of the matrix $Z_\ell(x,\lambda)$ (9) gives

$$(11)\quad z_{ij:\ell}=\delta_{ij}+\lambda^{-\beta}e^{\widetilde{Q}_{ij}(x,\lambda)}\phi_{ij:\ell}$$

where $\phi_{ij:\ell}\leq f(\beta)$, $(i=1,\ldots\tau;j\ 1,\ldots n)$. For a limiting function $z_{ij}(x,\lambda)$ we have

$$(12)\quad z_{ij}(x,\lambda)=\delta_{ij}+\lambda^{-\beta}e^{\widetilde{Q}_{ij}(x,\lambda)}\phi_{ij}(x,\lambda)\qquad |z_{ij}(x,\lambda)|<f(\beta).$$

Using (7) we have

$$(13)\quad y_{ij}(x,\lambda)=e^{Q_i(x,\lambda)}\left[\tau_{ij}(x,\lambda)+\lambda^{-\beta}{}_\beta\bar{\eta}_{ij}(x,\lambda)\right]$$

where ${}_\beta\bar{\eta}_{ij}(x,\lambda)<c(w)$. Taking $\beta=\beta(w)\leq t$ implies

$$(14)\quad y_{ij}(x,\lambda)\underset{\beta}{\sim}s_{ij}(x,\lambda).$$

LEMMA III. The system (2)

$$Y^{[1]}(x,\lambda)=Y(x,\lambda)B(x,\lambda)$$

has the formal matrix solution $S(x,\lambda)$ given by (5). Transforming equation (2) by (7) and solving the transformed equation (8) by means of product integrals, if the ordering

---

*Schlesinger, L., Vorlesungen uber Lineare Differentialgleichungen. Leipzig, 1908.

(1) is maintained for λ_i in R_i $(i=1,\ldots m)$ and x on (c,d), then

$$Y(x,\lambda) = Z(x,\lambda)T(x,\lambda)$$

will satisfy (2) and

$$y_{ij}(x,\lambda) \underset{\beta}{\sim} s_{ij}(x,\lambda) \quad (i=1,\ldots\tau_i)$$

where $\lim \beta(w) = \infty$. Similiar results are obtained if the inequality signs in (1) are reversed.

V. Fundamental Existence Theorem.

The equation (A) has a full set of formal solutions, $s_i(x,\lambda) = e^{\tilde{Q}_i(x,\lambda)} \sigma_i(x,\lambda)$ given by (1) §2. We limit the λ's to regions R in which the ordering (1) §4 is maintained and by Lemma 3, the system (B) has a linearly independent set of τ_i solutions ${}_iy_i^{[j-1]}(x,\lambda)$ such that ${}_iy_i^{[j-1]}(x,\lambda) \sim s_i^{[j-1]}(x,\lambda)$ $(i=1,\ldots\tau_i;$ $j=i,\ldots n)$. By means of a determinant we form the equation satisfied by these ${}_iy_i(x,\lambda)$

$$(1) \quad L_{\tau_i}(x,\lambda) = \sum_{k=0}^{\tau_i} {}_ia_{\tau_i-k}(x,\lambda)\, {}_iy^{[k]} = 0 \qquad ({}_ia_0 = 1).$$

By replacing the ${}_iy_i(x,\lambda)$ in (1) by the series $s_i(x,\lambda)$, we obtain the formal equation

$$(2) \quad L^*(x,\lambda;y) = \sum_{k=0}^{\tau_i} {}_i\alpha_{\tau_i-k}(x,\lambda)\, y^{[k]} = 0 \qquad ({}_i\alpha_0 = 1),$$

in which the ${}_i\alpha_i$'s are asymptotic to the ${}_ia_i$'s. Thus there exists an analytic factorization of (A)

$$(3) \quad L = L_n(x,\lambda;y) = L_{n-\tau_i}[z] = L_{n-\tau_i}[L_{\tau_i}(x,\lambda;y)] = 0$$

in which the coefficients are σ series. Lemma 3 can be applied to the equation of type (B) associated with $L_{n-\tau_i}(x,\lambda;z) = 0$. By means of the ordering relations (1) §4, and the Lemma, we obtain $\tau_i - \tau_i$ solutions of $L_{n-\tau_i}(z) = 0$,

given by ${}_1z_{\tau_i+i}(x,\lambda)$ $(i=1,\dots\tau_2-\tau_1)$ which are asymptotic to the corresponding $\tau_2-\tau_1$ solutions of the formal equation corresponding to (3).

In addition to the solutions ${}_1y_i(x,\lambda)$ $(i=1,\dots\tau_1)$ first obtained, we now have $\tau_2-\tau_1$ solutions ${}_2y_{\tau_1+i}(x,\lambda)$ $(i=1,\dots\tau_2-\tau_1)$ for equation (A) each satisfying the equation

$$(4)\quad L_{\tau_1}({}_2y) = {}_1z_{\tau_1+i}(x,\lambda) \qquad (1 \leqq i \leqq \tau_2-\tau_1).$$

A solution of (4) is given by

$$(5)\quad {}_2y_{\tau_1+i}(x,\lambda) = \sum_{r=1}^{\tau_1} {}_1y_r(x,\lambda) \int {}_1z_{\tau_1+i}(x,\lambda)\,\overline{{}_1y}_{\tau_1+r}(x,\lambda)\,dx$$

where the new ${}_2y_j(x,\lambda)$ satisfy the same conditions as the τ_1, ${}_1y_i(x,\lambda)$ formerly obtained.

Repeating the triple process; 1. Factorization, 2. Iteration, 3. Integration, $\nu-1$ additional times, we get a full set of solutions $(y_i(x,\lambda)$ $(i=1,\dots n)$, such that

$$(6)\quad y_i(x,\lambda) = {}_\nu y_i(x,\lambda) \sim e^{\widetilde{Q}_i(x,\lambda)}\,{}_\nu\mathcal{G}_i(x,\lambda) = s_i(x,\lambda).$$

We now state the Fundamental Existence Theorem;

THEOREM. Consider a region R in which the relations (1) § 4 hold. For $\lambda_1\dots\lambda_m$ in R and for x on (c,d), a sub-interval of (a,b) in which $q_o^{[n]}(x) \neq 0$, the equation (A)

$$L_n(x,\lambda) = \sum_{k=0}^{n} \lambda^{(n-k)+1}\, a_{n-k}(x,\lambda)\, y^{[k]}(x,\lambda) = 0$$

$$a_o(x,\lambda) = 1; \quad a_n(x,\lambda) \neq 0$$

where the coefficients satisfy the relations (A*), has a full set of solutions (6)

$$y_i(x,\lambda) \sim e^{\widetilde{Q}_i(x,\lambda)}\,\mathcal{G}_i(x,\lambda) = s_i(x,\lambda) \quad (i=1,\dots n)$$

where the $s_i(x,\lambda)$ are the formal solutions of the formal equation (2) § 2, corresponding to (A).

BIBLIOGRAPHY

1. Trjitzinsky, W. J.
Theory of linear differential equations containing a parameter.
Acta Mathematica, Vol. 67, (1936), pp. 1 - 50.

2. Hurd, C. C.
Asymptotic theory of linear differential equations containing two parameters.
Tohoku Mathematical Journal, Vol. 45, (Sept. 1938), pp. 58 - 68.

3. Trjitzinsky, W. J.
Analytic theory of linear differential equations.
Acta Mathematica, Vol. 62, (1934), pp. 167 - 226.

4. Hurd, C. C.
Asymptotic theory of linear differential equations singular in the variable of differentiation and in the parameter.
Tohoku Mathematical Journal, Vol. 44, (Mar. 1938), pp. 243 - 274.

5. Trjitzinsky, W. J.
Singular point problems in the theory of linear differential equations.
Bulletin, American Mathematical Society Vol. 44, (Apr. 1938), pp. 209 - 223.

VITA

Charles William Moran was born in Chicago, Illinois, June 15, 1907. He was educated in that city and graduated from Lake View High School in 1924. He attended Armour Institute of Technology in Chicago from 1924 to 1928. He held a position as an Architectural Engineer from 1926 to 1931. During 1931 he was employed by the U. S. Government, War Department, as an engineer. From 1931 to 1939 he taught at Lane Technical High School of Chicago while continuing his studies. He attended Lewis Institute in Chicago from 1932 to 1936, receiving the Bachelor of Science degree in 1933. He attended Loyola University and the University of Chicago from 1934 to 1939, receiving from the former institution the degree of Master of Arts in 1936, magna cum laude. He was a Lecturer in Mathematics at Loyola in 1936-37. During the summer of 1939 and the year 1939-1940 he attended the University of Illinois. He is a member of the Mathematical Association of America and the American Mathematical Society.